

Darboux transformations for a 6-point scheme ^{*}

M. Nieszporski ^{†‡}

February 8, 2008

Abstract

We introduce (binary) Darboux transformation for general differential equation of the second order in two independent variables. We present a discrete version of the transformation for a 6-point difference scheme. The scheme is appropriate to solving a hyperbolic type initial-boundary value problem. We discuss several reductions and specifications of the transformations as well as construction of other Darboux covariant schemes by means of existing ones. In particular we introduce a 10-point scheme which can be regarded as the discretization of self-adjoint hyperbolic equation.

Integrable systems, Jonas transformations, Moutard transformations

1 Introduction

One can observe increasing role of difference equations over the past few decades. Primarily efforts were undertaken to discretize differential equations so that not to lose the properties (e.g. symmetries) that differential equations exhibit. It turned out quickly that difference equations in many aspects are richer and more fundamental than their continuous counterparts (many interesting structures disappeared under a continuum limit) and difference equations started to be something more than equations mimetic differential equations. In the present work we encounter the essential differences between discrete and continuous mathematical structures once more.

The aim of this paper is to complete existing theory of Darboux transformations (or better to say Darboux–Moutard transformations [1, 2, 3] and Jonas

^{*}The initial stage of the work was supported by Polish KBN grants 2 PO3B 126 22 and 1 PO3B 017 28 while at the final stage (starting from 1st April 2005) the paper was supported solely by the European Community under a Marie Curie Intra-European Fellowship, contract no MEIF-CT-2005-011228.

[†]Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK e-mail: maciejun@maths.leeds.ac.uk, tel: +44 113 343 5149 fax: +44 113 343 5090

[‡]Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski ul. Hoża 74, 00-682 Warszawa, Poland e-mail: maciejun@fuw.edu.pl, tel: +48 22 621 77 57, fax: +48 22 622 45 08

transformations [4]) for differential equations and what more important to show the impact of the generalization on the theory of Darboux–Moutard transformations for difference equations.

The main idea of this paper is to start systematic surveys that can free theory of integrable systems from their strong dependence of coordinate systems (parametrization of surfaces), desirable by many physicists author spoke to. It is especially important for the so called difference geometry [5] since due to results of the paper in the case of difference equations one can compensate lack of possibility of change of independent variables $\tilde{x} = f(x, y)$, $\tilde{y} = g(x, y)$.

We recall that classical fundamental transformation by Jonas [4, 6] regarded as the most general Darboux transformation acts on the conjugate nets in projective space \mathbb{P}^n so in case when the net is two-dimensional it provide us with Darboux transformation for two-dimensional linear hyperbolic differential equation in canonical form (from now on, unless otherwise stated, small letters denote functions of real independent variables x and y and subscripts foregone by comma denote partial differentiation with respect to indicated variables)

$$\psi_{,xy} + w\psi_{,x} + z\psi_{,y} + f\psi = 0 \quad (1)$$

and the transformation is nothing but the spatial part of (binary) Darboux–Bäcklund transformation for two-component KP hierarchy. When the net is more than two-dimensional the Jonas fundamental transformation provides us with Darboux transformation for the set of compatible equations of the form (1) and serves as spatial part of binary Darboux–Bäcklund transformation for multicomponent KP hierarchies (in other words yields Bäcklund transformation for n-wave interaction equations called sometimes Darboux equations see e.g. [7, 8]).

The fundamental transformation has been successfully translated into the discrete language [9, 10, 11]. The discrete counterpart of conjugate nets are quadrilateral lattices in \mathbb{P}^n governed by system of equations of the type (unless otherwise stated in the whole paper capital letters denote functions of two discrete variables m and n ($(m, n) \in \mathbb{Z}^2$), Δ_m and Δ_n denotes forward difference operators $\Delta_m \Psi := \Psi_{m+1, n} - \Psi$ and $\Delta_n \Psi := \Psi_{m, n+1} - \Psi$ while Δ_{-m} and Δ_{-n} denotes backward difference operators $\Delta_{-m} \Psi := \Psi_{m-1, n} - \Psi$ and $\Delta_{-n} \Psi := \Psi_{m, n-1} - \Psi$, note we identify $\Psi \equiv \Psi_{m, n}$ and in the whole paper we apply this convention)

$$\Delta_m \Delta_n \Psi + A \Delta_m \Psi + B \Delta_n \Psi + C \Psi = 0 \quad (2)$$

That is why in recent years notion of integrability of discrete (difference) equations was often related to the planarity - 4-point schemes were the building blocks of the theory.

Generalization to the so called quad-graphs (planar graphs), still objects with incorporated planarity, appeared only recently [12, 13, 14, 15].

In the present paper we show that planarity is not crucial from the point of view of integrable systems. It is remarkable but only example of more general theory with a 6-point difference scheme and 7-point self-adjoint difference

scheme as building blocks of the discrete theory of Darboux-Moutard transformations.

It turns out that the general differential equation of the second order in two independent variables

$$(a\psi_{,x} + c\psi_{,y})_{,x} + (c\psi_{,x} + b\psi_{,y})_{,y} + w\psi_{,x} + z\psi_{,y} - f\psi = 0$$

is covariant under a Darboux transformations (section 3) so conjugate nets are no longer of key importance. On the discrete level it reflects in the fact that one can generalize 4-point scheme to a 6-point scheme (see Fig. 1)

$$A\Psi_{m+2,n} + B\Psi_{m,n+2} + 2C\Psi_{m+1,n+1} + G\Psi_{m+1,n} + H\Psi_{m,n+1} = F\Psi$$

and quadrilateral lattices cease to be the master object of study in favour of triangular lattices with the 6-point scheme defined on them (section 4).

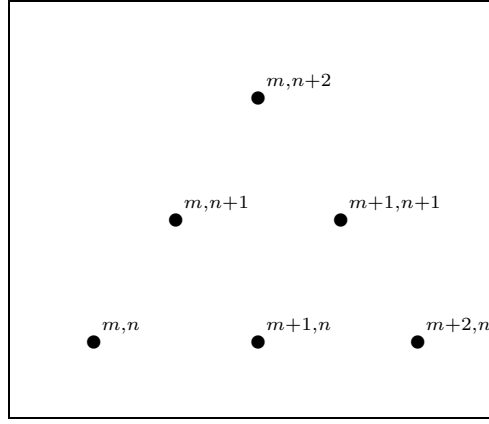


Figure 1: The 6-point scheme

Moutard transformation for the 7-point self-adjoint scheme (see Fig. 2)

$$\mathcal{A}_{m+1,n} N_{m+1,n} + \mathcal{A} N_{m-1,n} + \mathcal{B}_{m,n+1} N_{m,n+1} + \mathcal{B} N_{m,n-1} + \mathcal{C}_{m+1,n} N_{m+1,n-1} + \mathcal{C}_{m,n+1} N_{m-1,n+1} = \mathcal{F} N,$$

an example of equation given on a star (cross), has been derived in the paper [16] (see also subsection 7.2) and that is why we concentrate here mainly on the 6-point scheme. However, the present paper is thought to provide a brief overview on the topic of discretizations of 2D second order differential equations that are covariant under a Darboux transformations and the reader can find in the closing section 8 references to articles on integrable aspects of equations given on stars. At the moment we only underline that choice of difference scheme restricts sorts of initial-boundary value problems one can solve by means of the scheme. So it is important to indicate first on sort of initial-boundary conditions one would like to solve and then consider only the schemes that allow to solve

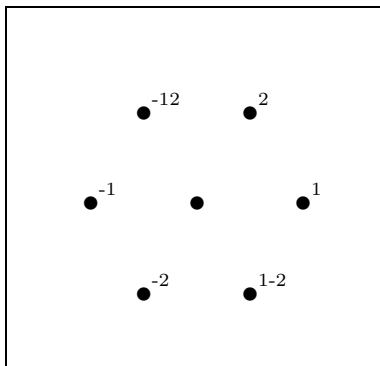


Figure 2: 7-point scheme

the initial-boundary value problem. We start the paper with description of a well like initial-boundary value problem (section 2) we have in mind while 6-point scheme is considered. In turn the 7-point scheme is not proper to solve the initial-boundary value problem (as we know from numerical analysis schemes on stars are suitable to solve numerically Dirichlet boundary value problem for an elliptic differential equation, see e.g. [17]) but one can construct a proper scheme using the 7-point scheme (see subsection 7.3).

In this paper we discuss also how the general Darboux transformation can be reduced or specified. We take the stand that introduction of novel terminology (such as specification) is necessary to discern procedures we deal with. We begin the discussion of reductions and specifications from the continuous case (section 5). Firstly, we consider the Moutard reduction (subsection 5.1) which is very classical construction [1] but to the best of our knowledge in full generality was given only recently [16]. Secondly, we discuss specifications (subsection 5.2) and this part is (stands to reasons) new, specification to hitherto considered "conjugate" case or its elliptic counterpart are just examples of such procedure. Thirdly, we discuss two convenient gauge specifications (subsection 5.3) the transformations can be written in. Finally we discuss reductions and specifications in the discrete case.

We start from gauge specifications (subsection 6.1) and specifications (subsection 6.2). We are not able to show reduction of the general 6-point scheme that leads to transformation of the Moutard type. Therefore we recall all results related to the topic first (section 7) and then all we are able to do is to introduce a 10-point scheme which is appropriate for solving defined in section 2 initial boundary value problem and can be regarded as a discretization of self-adjoint differential equation (section 7.3).

We would like to stress once more that although we deal in the paper with linear equations only, the existence of Darboux-Moutard transformations makes this paper especially important for the theory of nonlinear integrable systems.

2 Well like initial-boundary value problem for 6-point scheme

In the present paper we pay special attention to difference schemes that allows to solve the following initial boundary value problem. We prescribe function $\Psi(m, n)$ in the following points of the domain (see Fig. 3)

- initial conditions

$$\{(m, n) \in \mathbb{T} \mid m + n = 0 \vee m + n = 1\}$$

- boundary conditions

$$\{(m, n) \in \mathbb{T} \mid (m = s - p_s \wedge n = p_s) \vee (m = s - p_s \wedge n = p_s), s = 2, 3, 4, \dots\}$$

where \mathbb{T} denotes regular triangular lattice, p_s and q_s are functions $\mathbb{N} \setminus \{1\} \ni s \mapsto p_s \in \mathbb{Z}$ such that $\forall s \in \mathbb{N} \setminus \{1\} p_s < q_s$

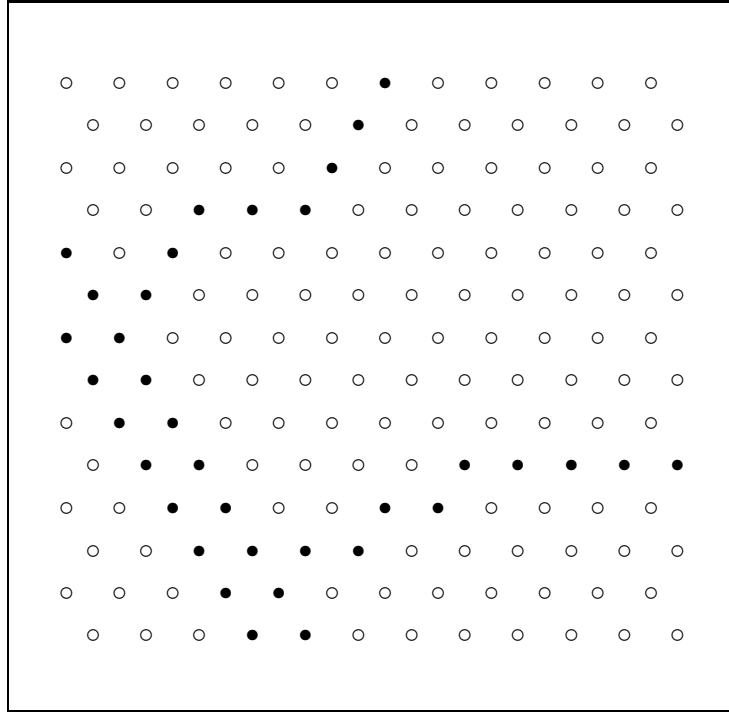


Figure 3: Initial-boundary value problem. The initial values at points of two neighbouring straight line are given as well as two boundary conditions (black points).

We concentrate in the paper mainly on the schemes that allow to find solution uniquely at least in the "upper half-plane" $\{(m, n) \in \mathbb{T} \mid m + n \geq 0\}$ of the lattice.

For instance in the case of the 6-point scheme if the following conditions are satisfied

1)

$$\forall (m, n) \in \mathbb{N} \times \mathbb{N}, A_{m,n} \neq 0, B_{m,n} \neq 0, F_{m,n} \neq 0$$

2)

$$\forall s \in \mathbb{N} \setminus \{1\}$$

the matrices

$$\begin{bmatrix} 2C_{s-p_s-2,p_s} & B_{s-p_s-2,p_s} & 0 & \cdots & 0 \\ A_{s-p_s-3,p_s+1} & 2C_{s-p_s-3,p_s+1} & B_{s-p_s-3,p_s+1} & 0 & 0 \\ 0 & & & \ddots & \vdots \\ \vdots & \ddots & & & 0 \\ 0 & \cdots & 0 & A_{s-q_s+1,q_s-1} & 2C_{s-q_s+1,q_s-1} & B_{s-q_s+1,q_s-1} \\ 0 & & \cdots & 0 & A_{s-q_s,q_s-2} & 2C_{s-q_s,q_s-2} \end{bmatrix}$$

have non-vanishing determinant, then all the values at white points can be found uniquely. Similar result can be obtain for the 10-point scheme (49) with the only essential difference that the solution can be found uniquely only in upper half-plane.

3 Darboux transformations for 2D second order differential equation

It is a basic observation that map $\psi^t \mapsto \bar{\psi}^t$ given by

$$\bar{\psi}^t_{,x} = \delta \psi^t_{,x} + \beta \psi^t_{,y} \quad \bar{\psi}^t_{,y} = -\alpha \psi^t_{,x} - \gamma \psi^t_{,y}, \quad (3)$$

where α, β, γ and δ are \mathcal{C}^1 real functions (of independent variables x and y with an open, simply connected subset \mathcal{D} of \mathbb{R}^2 as a domain) such that $\forall (x, y) \in \mathcal{D}$, $\gamma\delta - \alpha\beta \neq 0$ and $\alpha^2 + \beta^2 + (\gamma + \delta)^2 \neq 0$, is an invertible map between solution spaces of two differential equations of second order in two independent variables. Indeed, the compatibility condition of (3), which ensures existence of $\bar{\psi}^t$ function, reads

$$\mathcal{L}^t \psi^t = 0 \quad (4)$$

$$\mathcal{L}^t := \alpha \partial_x^2 + \beta \partial_y^2 + (\gamma + \delta) \partial_x \partial_y + (\alpha_{,x} + \delta_{,y}) \partial_x + (\beta_{,y} + \gamma_{,x}) \partial_y.$$

Obviously $\bar{\psi}^t$ satisfies equation of the same type but with bared coefficients

$$\bar{\alpha} = \frac{\alpha}{\gamma\delta - \alpha\beta}, \quad \bar{\beta} = \frac{\beta}{\gamma\delta - \alpha\beta}, \quad \bar{\gamma} = \frac{\delta}{\gamma\delta - \alpha\beta}, \quad \bar{\delta} = \frac{\gamma}{\gamma\delta - \alpha\beta}. \quad (5)$$

As we shall see every second order equation in two independent variables

$$\mathcal{L}^f \psi = 0 \quad (6)$$

$$\mathcal{L}^f := a \partial_x^2 + b \partial_y^2 + 2c \partial_x \partial_y + (a_{,x} + c_{,y} + w) \partial_x + (b_{,y} + c_{,x} + z) \partial_y - f$$

can be transformed into the form (4) through a gauge transformation

$$\mathcal{L}^f \mapsto \mathcal{L} := \hat{\phi} \mathcal{L}^f \hat{\theta} \quad (7)$$

where $\hat{\phi}$ and $\hat{\theta}$ are operators of multiplying by function $\phi(x, y)$ and $\theta(x, y)$ respectively, which we are going to determine now. We will call operator \mathcal{L}^t *elementary transformable form* of the second order differential operator.

Indeed, request that operator \mathcal{L} defined in eq. (7) is of the elementary transformable form \mathcal{L}^t (4) gives

$$\alpha = \phi \theta a \quad \beta = \phi \theta b \quad \gamma + \delta = 2\phi \theta c \quad (8)$$

$$\mathcal{L}^f \theta = 0 \quad (9)$$

$$\begin{aligned} \alpha_{,x} + \delta_{,y} &= [(a\theta)_{,x} + (c\theta)_{,y} + a\theta_{,x} + c\theta_{,y} + w\theta]\phi, \\ \beta_{,y} + \gamma_{,x} &= [(c\theta)_{,x} + (b\theta)_{,y} + c\theta_{,x} + b\theta_{,y} + z\theta]\phi. \end{aligned} \quad (10)$$

On introducing auxiliary function p

$$p := \frac{1}{2\phi\theta}(\delta - \gamma) \quad (11)$$

and treating eqs (8), (11) as the definition of functions α , β , γ and δ and eliminating these functions from equations (10) one can rewrite equations (10) in the form

$$\begin{aligned} (\theta\phi p)_{,y} &= \phi^2 [a(\frac{\theta}{\phi})_{,x} + c(\frac{\theta}{\phi})_{,y} + w\frac{\theta}{\phi}] \\ (\theta\phi p)_{,x} &= - \phi^2 [c(\frac{\theta}{\phi})_{,x} + b(\frac{\theta}{\phi})_{,y} + z\frac{\theta}{\phi}] \end{aligned} \quad (12)$$

Function p exists provided that

$$\phi \mathcal{L}^f \theta - \theta (\mathcal{L}^f)^\dagger \phi = 0 \quad (13)$$

Where $(\mathcal{L}^f)^\dagger$ denotes the operator formally adjoint to the operator \mathcal{L}^f

$$(\mathcal{L}^f)^\dagger := \partial_x(a\partial_x + c\partial_y - w) + \partial_y(b\partial_y + c\partial_x - z) - f \quad (14)$$

Taking into account (9) and (13) we come to the theorem

Theorem 1 *Gauge transformation (7) makes from arbitrary 2D second order operator \mathcal{L}^f an operator in elementary transformable form iff*

$$\mathcal{L}^f \theta = 0 \quad \text{and} \quad (\mathcal{L}^f)^\dagger \phi = 0 \quad (15)$$

It turns out that just presented considerations lead to Darboux transformations for the 2D second order operator \mathcal{L}^f . Namely we have conclusion

Conclusion 1 (Darboux transformations) *We assume that θ and ϕ are \mathcal{C}^2 class functions satisfying conditions (15), function p is given by formulae (12), r and s are arbitrary (of class \mathcal{C}^2) functions, and function d given by $d := (p^2 - c^2 + ab)\phi\theta$ obeys condition $\forall(x, y) \in \mathcal{D}; d \neq 0$. Then the map $\psi \mapsto \bar{\psi}$ given by*

$$\begin{bmatrix} (s\bar{\psi})_{,x} \\ (s\bar{\psi})_{,y} \end{bmatrix} = \phi\theta \begin{bmatrix} p+c & b \\ -a & p-c \end{bmatrix} \begin{bmatrix} \left(\frac{\psi}{\theta}\right)_{,x} \\ \left(\frac{\psi}{\theta}\right)_{,y} \end{bmatrix} \quad (16)$$

is the map from solution space of equation (6) to the solution space of the equation of the same form but with the new coefficients

$$\bar{\mathcal{L}}\bar{\psi} = 0$$

$$\bar{\mathcal{L}} := \bar{a}\partial_x^2 + \bar{b}\partial_y^2 + 2\bar{c}\partial_x\partial_y + (\bar{a}_{,x} + \bar{c}_{,y} + \bar{w})\partial_x + (\bar{c}_{,x} + \bar{b}_{,y} + \bar{z})\partial_y - \bar{f} \quad (17)$$

where the coefficients of (17) are related to coefficients of (6) by

$$\bar{a} = \frac{asr}{d}, \quad \bar{b} = \frac{bsr}{d}, \quad \bar{c} = \frac{csr}{d},$$

$$\bar{w} = \left[\frac{a}{d}\left(\frac{s}{r}\right)_{,x} + \frac{c}{d}\left(\frac{s}{r}\right)_{,y} - \left(\frac{p}{d}\right)_{,y}\frac{s}{r}\right]r^2, \quad \bar{z} = \left[\frac{b}{d}\left(\frac{s}{r}\right)_{,y} + \frac{c}{d}\left(\frac{s}{r}\right)_{,x} + \left(\frac{p}{d}\right)_{,x}\frac{s}{r}\right]r^2. \quad (18)$$

$$\bar{f} = \{-[\frac{a}{d}s_x + \frac{c+p}{d}s_y]_x - [\frac{b}{d}s_y + \frac{c-p}{d}s_x]_y\}r$$

4 6-point scheme and its Darboux transformations

One can repeat considerations from the previous section in the discrete case. Indeed, starting from pair of equations

$$\Delta_m \bar{\Psi}^t = \delta \Delta_m \Psi^t + \beta \Delta_n \Psi^t \quad \Delta_n \bar{\Psi}^t = -\alpha \Delta_m \Psi^t - \gamma \Delta_n \Psi^t \quad (19)$$

(where the functions α , β , γ and δ are functions of discrete variables m and n) and writing down their compatibility condition

$$\begin{aligned} & \alpha_{m+1,n} \Psi_{m+2,n}^t + \beta_{m,n+1} \Psi_{m,n+2}^t + (\gamma_{m+1,n} + \delta_{m,n+1}) \Psi_{m+1,n+1}^t - \\ & (\alpha_{m+1,n} + \alpha + \gamma_{m+1,n} + \delta) \Psi_{m+1,n}^t - (\beta_{m,n+1} + \beta + \gamma + \delta_{m,n+1}) \Psi_{m,n+1}^t + \\ & (\alpha + \beta + \gamma + \delta) \Psi^t = 0 \end{aligned} \quad (20)$$

which is a 6-point scheme. One can ask if it is possible to transform the general 6-point scheme of this type

$$A\Psi_{m+2,n} + B\Psi_{m,n+2} + 2C\Psi_{m+1,n+1} + G\Psi_{m+1,n} + H\Psi_{m,n+1} = F\Psi \quad (21)$$

$$L^F \Psi = 0$$

into the form (20) via a gauge transformation

$$L^F \mapsto L := \hat{\Phi} L^F \hat{\Theta} \quad (22)$$

only? The answer is positive and we will be calling equation of type (20) a 6-point scheme in *elementary transformable form*.

Theorem 2 Gauge transformation (22) makes from the 6-point scheme (21) L^f an operator in elementary transformable form L^t iff the function Θ satisfies eq. (21)

$$A\Theta_{m+2,n} + B\Theta_{m,n+2} + 2C\Theta_{m+1,n+1} + G\Theta_{m+1,n} + H\Theta_{m,n+1} = F\Theta \quad (23)$$

while function Φ is a solution of the equation formally adjoint to eq. (21)

$$\begin{aligned} A_{m-2,n}\Phi_{m-2,n} + B_{m,n-2}\Phi_{m,n-2} + 2C_{m-1,n-1}\Phi_{m-1,n-1} + \\ G_{m-1,n}\Phi_{m-1,n} + H_{m,n-1}\Phi_{m,n-1} = F\Phi \end{aligned} \quad (24)$$

Then the functions α , β , γ and δ in eq. (20) are given by

$$\begin{aligned} \alpha &= A_{m-1,n}\Phi_{m-1,n}\Theta_{m+1,n} \\ \beta &= B_{m,n-1}\Phi_{m,n-1}\Theta_{m,n+1} \\ \gamma &= (C_{m-1,n} - P_{m-1,n})\Phi_{m-1,n}\Theta_{m,n+1} \\ \delta &= (C_{m,n-1} + P_{m,n-1})\Phi_{m,n-1}\Theta_{m+1,n} \end{aligned} \quad (25)$$

where P is an auxiliary function defined by

$$\begin{aligned} \Delta_{-m}(\Phi\Theta_{m+1,n+1}P) &= -(B_{m,n-1}\Phi_{m,n-1}\Theta_{m,n+1} + B\Phi\Theta_{m,n+2} + \\ &\quad C_{m-1,n}\Phi_{m-1,n}\Theta_{m,n+1} + C\Phi\Theta_{m+1,n+1} + H\Phi\Theta_{m,n+1}) \\ \Delta_{-n}(\Phi\Theta_{m+1,n+1}P) &= A_{m-1,n}\Phi_{m-1,n}\Theta_{m+1,n} + A\Phi\Theta_{m+2,n} + \\ &\quad C_{m,n-1}\Phi_{m,n-1}\Theta_{m+1,n} + C\Phi\Theta_{m+1,n+1} + G\Phi\Theta_{m+1,n} \end{aligned} \quad (26)$$

We have the conclusion

Conclusion 2 (Darboux transformations for the 6-point scheme)

The map $\Psi \mapsto \bar{\Psi}$ given by

$$\begin{aligned} \begin{bmatrix} \Delta_m(S\bar{\Psi}) \\ \Delta_n(S\bar{\Psi}) \end{bmatrix} &= \\ \begin{bmatrix} (C_{m,n-1} + P_{m,n-1})\Phi_{m,n-1}\Theta_{m+1,n} & B_{m,n-1}\Phi_{m,n-1}\Theta_{m,n+1} \\ -A_{m-1,n}\Phi_{m-1,n}\Theta_{m+1,n} & (P_{m-1,n} - C_{m-1,n})\Phi_{m-1,n}\Theta_{m,n+1} \end{bmatrix} \begin{bmatrix} \Delta_m\left(\frac{\Psi}{\Theta}\right) \\ \Delta_n\left(\frac{\Psi}{\Theta}\right) \end{bmatrix} \end{aligned} \quad (27)$$

where function Θ satisfies eq. (21) while function Φ is a solution of the eq. (24) and P is defined via equations (26), is the map from solution space of the equation (21) to the solution space of the equation of the same form with novel

(bared) coefficients related to the old ones via

$$\begin{aligned}
\bar{A} &= \frac{RS_{m+2,n}\Phi\Theta_{m+2,n}}{D_{m+1,n}}A, & \bar{B} &= \frac{RS_{m,n+2}\Phi\Theta_{m,n+2}}{D_{m,n+1}}B, & \bar{F} &= \frac{RS\Phi\Theta}{D}F, \\
\frac{2\bar{C}}{RS_{m+1,n+1}} &= \frac{\Theta_{m+2,n}\Phi_{m+1,n-1}(C+P)_{m+1,n-1}}{D_{m+1,n}} + \frac{\Theta_{m,n+2}\Phi_{m-1,n+1}(C-P)_{m-1,n+1}}{D_{m,n+1}}, \\
\frac{\bar{G}}{RS_{m+1,n}} &= -\frac{\Theta_{m+2,n}\Phi_{m+1,n-1}(C+P)_{m+1,n-1} + \Theta_{m+2,n}\Phi A}{D_{m+1,n}} - \\
&\quad + \frac{\Theta_{m,n+1}\Phi_{m-1,n}(C-P)_{m-1,n} + \Theta_{m+1,n}\Phi_{m-1,n}A_{m-1,n}}{D}, \\
\frac{\bar{H}}{RS_{m,n+1}} &= -\frac{\Theta_{m,n+2}\Phi_{m-1,n+1}(C-P)_{m-1,n+1} + \Theta_{m,n+2}\Phi B}{D_{m,n+1}} - \\
&\quad + \frac{\Theta_{m+1,n}\Phi_{m,n-1}(C+P)_{m,n-1} + \Theta_{m,n+1}\Phi_{m,n-1}B_{m,n-1}}{D},
\end{aligned} \tag{28}$$

where R and S are arbitrary non-vanishing functions, while D is a function given by

$D = [(P_{m-1,n} - C_{m-1,n})(P_{m,n-1} + C_{m,n-1}) + A_{m-1,n}B_{m,n-1}] \Theta_{m+1,n} \Theta_{m,n+1} \Phi_{m-1,n} \Phi_{m,n-1}$ and is assumed not to vanish on the whole lattice.

5 Gauge specifications, specifications and reductions

One can reduce (or gauge or specify) the Darboux transformation in such a way that it maps between solution space of restricted class of equations. Assume first that matrix coefficients in eq. (16) and coefficients of its inverse obey the same linear constraint

$$\begin{aligned}
a^{11}(p+c) + a^{12}b - a^{21}a + a^{22}(p-c) &= 0 \\
a^{11}(p-c) - a^{12}b + a^{21}a + a^{22}(p+c) &= 0
\end{aligned} \tag{29}$$

where a^{ij} are given functions of x and y . From (29) we infer

$$(a^{11} + a^{22})p = 0$$

and will discuss two cases $p = 0$ and $a^{11} + a^{22} = 0$ separately.

5.1 Moutard reduction ($p = 0$), reductions

We have $p = 0$ and $(a^{11} - a^{22})c + a^{12}b - a^{21}a = 0$ to be satisfied. The later constraint can be satisfied if one take $a^{11} = a^{22}$, $a^{12} = 0$ and $a^{21} = 0$. To satisfy the equations (12) in the presence of condition $p = 0$ it is enough to put $\phi = \theta$ and $w = 0 = z$ (i.e. demand that operator is formally self-adjoint). Moreover the functions r and s are no longer arbitrary, they must obey constraint $\frac{r}{s} = \text{const.}$ As a result we obtain a transformation for formally self-adjoint equations which

are usually refereed to (in the case $a = 0 = b$, $c = \frac{1}{2}$, $s = r = \frac{1}{2}\theta$) as Moutard transformation.

The procedure that impose the constraints on transformation's data ϕ and θ ($\theta = \phi$ in the Moutard case) we call reduction of the transformation.

5.2 Specifications $a^{11} + a^{22} = 0$

The reduction is not the only procedure we have to our disposal. If we take $a^{11} = -a^{22}$ we have the constraint $2a^{11}c + a^{12}b - a^{21}a = 0$ to be satisfied. Two examples are

- a) $a = 0 = b$, $a^{11} = 0$ and $c = 1$ which is nothing but specification to "conjugate" case
- b) $c = 0$, $a^{12} = 0 = a^{21}$ (with option for deeper specification $a = \pm b$)

In those cases we only specify (specialize) the operator not affecting the transformation data.

5.3 Gauge specifications, affine form, elementary transformable form

The idea not to consider the operator itself but the equivalence classes with respect to the gauge goes back to Laplace and Darboux papers [18]. One can then develop theory in gauge independent language or choose a gauge appropriate to ones needs. We concentrate on the later procedure We take two arbitrary functions θ^0, ϕ^0 of \mathcal{C}^2 class. Then operator

$$L^g := \hat{\phi}^0 L^f \hat{\theta}^0$$

has coefficients

$$(a^g, b^g, c^g, f^g) = \theta^0 \phi^0 (a, b, c, L^f \theta^0)$$

$$w^g = \theta^0 \phi^0 w - (\theta^0)^2 \left(\frac{\phi^0}{\theta^0}\right)_{,x} a - (\theta^0)^2 \left(\frac{\phi^0}{\theta^0}\right)_{,y} c$$

$$z^g = \theta^0 \phi^0 z - (\theta^0)^2 \left(\frac{\phi^0}{\theta^0}\right)_{,y} b - (\theta^0)^2 \left(\frac{\phi^0}{\theta^0}\right)_{,x} c$$

and operator adjoint to L^g is

$$(L^g)^\dagger := \hat{\theta}^0 L^f \hat{\phi}^0$$

If in addition we define

$$\psi^g = \frac{\psi}{\theta_0} \quad \phi^g = \frac{\phi}{\phi_0} \quad \theta^g = \frac{\theta}{\theta_0} \quad (30)$$

then the form of the transformation remains unaltered

$$\begin{aligned} (\theta^g \phi^g p^g)_{,y} &= (\phi^g)^2 [a^g \left(\frac{\theta^g}{\phi^g}\right)_{,x} + c^g \left(\frac{\theta^g}{\phi^g}\right)_{,y} - w^g \frac{\theta^g}{\phi^g}] \\ (\theta^g \phi^g p^g)_{,x} &= - (\phi^g)^2 [c^g \left(\frac{\theta^g}{\phi^g}\right)_{,x} + b^g \left(\frac{\theta^g}{\phi^g}\right)_{,y} - z^g \frac{\theta^g}{\phi^g}] \end{aligned} \quad (31)$$

$$\begin{bmatrix} (s^g \bar{\psi}^g)_{,x} \\ (s^g \bar{\psi}^g)_{,y} \end{bmatrix} = \phi^g \theta^g \begin{bmatrix} p^g + c^g & b^g \\ -a^g & p^g - c^g \end{bmatrix} \begin{bmatrix} \left(\frac{\psi^g}{\theta^g} \right)_{,x} \\ \left(\frac{\psi^g}{\theta^g} \right)_{,y} \end{bmatrix} \quad (32)$$

We indicate on two convenient gauges:

1) Elementary transformable gauge

$$L^f \theta^0 = 0 \quad (L^g)^\dagger \phi^0 = 0$$

This gauge specification of the transformation to an elementary transformable form i.e., as we know from theorem 1, conditions

$$f^g = 0 \quad w^g_{,x} + z^g_{,y} = 0$$

hold. Moreover the operator $(L^g)^\dagger$ is in elementary transformable form as well. The functions s^g and r^g are no longer arbitrary. To assure the transformed (bared) equation be in the elementary transformable form it is enough to put the functions s^g and r^g to be constant.

2) Affine gauge is

$$L^f \theta^0 = 0$$

so only

$$f^A = 0$$

holds. Now it is enough to put s^A to be a constant to obtain bared equation in affine form. Further convenient gauge specification is possible by putting

$$\phi^0 = \theta^0$$

Then the operator L^g has the coefficients

$$(a^A, b^A, c^A, w^A, z^A, f^A) = (\theta_0)^2 (a, b, c, w, z, 0)$$

6 Specifications, discrete case

6.1 Gauge specifications

In the discrete one can specify the gauge as well. Namely, we take two arbitrary functions Θ^0 Φ^0 Then operator

$$L^g := \hat{\Phi}^0 L^f \hat{\Theta}^0$$

has coefficients

$$(A^g, B^g, C^g, G^g, H^g, F^g) = \Phi^0 (A\Theta_{m+2,n}^0, B\Theta_{m,n+2}^0, C\Theta_{m+1,n+1}^0, G\Theta_{m+1,n}^0, H\Theta_{m,n+1}^0, F\Theta^0)$$

The two examples of convenient gauges are

1) Affine gauge which we obtain demanding that Θ^0 satisfies the equation

$$L^f \Theta^0 = 0 \quad (33)$$

Then coefficients of operator L^g obey constraint

$$A^g + B^g + 2C^g + G^g + H^g - F^g = 0$$

If one puts $S = \text{const}$ then the constrain is preserved under the Darboux transformation.

2) We demand that Φ^0 satisfies the equation

$$(L^f)^\dagger \Phi^0 = 0 \quad (34)$$

Then coefficients of operator L^g obey constraint

$$A_{m-1,n+1}^g + B_{m+1,n-1}^g + 2C^g + G_{m,n+1}^g + H_{m+1,n}^g - F_{m+1,n+1}^g = 0$$

If one puts $R = \text{const}$ then the constrain is preserved under the Darboux transformation.

If we apply the conditions (33) and (34) together and demand $R = 0 = S$ then we obtain transformation between elementary invertible forms of the 6-point scheme.

6.2 Specifications, quadrilateral lattices, 3-point scheme

A glance at transformation laws of the coefficients of the 6-point scheme (28) provide us with conclusions

A) Both constraint $A = 0$, $B = 0$ and $F = 0$ are preserved under the Darboux transformation (27). Constraint $A = 0 = B$ (or alternatively $A = 0 = F$ or $B = 0 = F$) is specification to the celebrated 4-point scheme i.e. to quadrilateral lattice case subject of study of many papers (we confine ourselves to citing articles where Darboux transformations are considered)

- Quadrilateral Lattices (Jonas fundamental transformations) [10, 11, 20, 21, 22]
- Circular Lattices and Quadratic Reductions (Ribaucour type transformations) [25, 26, 27, 28]
- Moutard type transformations [23, 24]
- Symmetric (Goursat) type transformations [29, 30]

Initial boundary value problem suitable for this scheme is no longer of the type mentioned in section 2.

B) Constraint $C = 0$ is not preserved under the Darboux transformation (27). So we have not got discretization of specification b) from the section 5.2.

C) If we impose $F = 0$ together with $A = 0 = B$ we obtain Darboux transformation for a 3-point scheme which corresponds to the continuous degenerated case $a^2 + b^2 + c^2 = 0$.

7 Discrete Moutard case

The question arises can one find difference scheme which is appropriate to solve initial boundary conditions we presented in section 2 and can be regarded as a discretization of Moutard reduction? We were not able to find such a reduction of 6-point scheme and we are not able to give a satisfactory non-existence theorem of such a 6-point scheme. Instead we construct the Darboux covariant 10-point scheme that can be regarded as a discrete Moutard equation and is suitable to solve initial boundary condition we described in section 2.

In this section we firstly recall (for completeness of the paper) known results i.e. the discrete Moutard equation and its adjoint which were known only for the quadrilateral specification, so far. Secondly we recall self-adjoint 7-point scheme which is not proper to solve mentioned in the section 2 initial-boundary value problem. Finally we introduce a discretization of general Moutard reduction which is not just a reduction of 6-point scheme but arise from such reformulation of 7-point scheme that allows for solving the initial-boundary value problem.

7.1 Discrete Moutard equations

On putting $A = 0 = B$ (quadrilateral specification), $2C = -F$ and $G = H =: MF$ the equations (21), (23) and (24) take respectively form

$$F(\Psi_{m+1,n+1} + \Psi) = G(\Psi_{m+1,n} + \Psi_{m,n+1}) \quad (35)$$

$$F(\Theta_{m+1,n+1} + \Theta) = G(\Theta_{m+1,n} + \Theta_{m,n+1}) \quad (36)$$

$$F_{m-1,n-1}\Phi_{m-1,n-1} + F\Phi = G_{m-1,n}\Phi_{m-1,n} + G_{m,n-1}\Phi_{m,n-1} \quad (37)$$

The crucial observation is: if the function Θ satisfies equation (36) then the function Φ given by

$$\Phi := \frac{1}{F}(\Theta_{m+1,n} + \Theta_{m,n+1}) \quad (38)$$

satisfies equation (37) [24]. If we put

$$2P = \frac{\Theta_{m+1,n} - \Theta_{m,n+1}}{\Phi}$$

then equations (26) will be automatically satisfied. If in addition we put $S = \Theta$ then Darboux transformation (27) takes form

$$\begin{aligned} \Delta_n \Theta \bar{\Psi} &= \Theta \Theta_{m,n+1} \Delta_n \frac{\Psi}{\Theta} \\ \Delta_m \Theta \bar{\Psi} &= -\Theta \Theta_{m+1,n} \Delta_m \frac{\Psi}{\Theta} \end{aligned} \quad (39)$$

which is the discrete Moutard transformation given by Nimmo and Schief [23]. Function $\bar{\Psi}$ satisfies the equation

$$\bar{\Psi}_{m+1,n+1} + \bar{\Psi} = \bar{M}(\bar{\Psi}_{m+1,n} + \bar{\Psi}_{m,n+1}) \quad (40)$$

where

$$\bar{M} = \frac{\frac{1}{\Theta_{m+1,n}} + \frac{1}{\Theta_{m,n+1}}}{\frac{1}{\Theta_{m+1,n+1}} + \frac{1}{\Theta}} \quad (41)$$

Equations (36) and (37) are not the same so we cannot say that we have derived self-adjoint reduction. That is why we'd prefer to call the reduction Moutard reduction rather than self-adjoint reduction.

7.2 7-point self-adjoint scheme and its 5-point specification

To made the paper self-contained we derive in different manner the results contained in [16]. In the discrete case one can write

$$\Delta_m \bar{N}^t = \delta \Delta_{-m} N^t + \beta \Delta_{-n} N^t \quad \Delta_n \bar{N}^t = -\alpha \Delta_{-m} N^t - \gamma \Delta_{-n} N^t \quad (42)$$

so instead of forward difference operators on the right side of equations (19) we have introduced backward difference operators in (42). The compatibility condition provide us with an elementary transformable 7-point scheme

$$\Delta_n(\delta \Delta_{-m} N^t + \beta \Delta_{-n} N^t) + \Delta_m(\alpha \Delta_{-m} N^t + \gamma \Delta_{-n} N^t) = 0$$

and function \bar{N} satisfies an elementary transformable 7-point scheme as well. The question arises can one transform the general 7-point scheme

$$\mathcal{A}_{m+1,n} \Psi_{m+1,n} + \mathcal{G} \Psi_{m-1,n} + \mathcal{B}_{m,n+1} \Psi_{m,n+1} + \mathcal{H} \Psi_{m,n-1} + \mathcal{C}_{m+1,n} \Psi_{m+1,n-1} + \mathcal{D}_{m,n+1} \Psi_{m-1,n+1} = \mathcal{F} \Psi$$

to an elementary transformable 7-point scheme by a gauge transformation only? The answer is negative this time because apart from equations

$$\begin{aligned} \gamma &= \mathcal{G}_{m-1,n} \Phi_{m-1,n} \Theta_{m,n-1}, \quad \delta = \mathcal{H}_{m,n-1} \Phi_{m,n-1} \Theta_{m-1,n}, \\ \alpha &= -\mathcal{A} \Phi_{m-1,n} \Theta - \mathcal{G}_{m-1,n} \Phi_{m-1,n} \Theta_{m,n-1} \\ \beta &= -\mathcal{B} \Phi_{m,n-1} \Theta - \mathcal{H}_{m,n-1} \Phi_{m,n-1} \Theta_{m-1,n} \end{aligned} \quad (43)$$

one has to satisfy three equations

$$\begin{aligned} \mathcal{G} \Phi \Theta_{m-1,n} + \mathcal{D} \Phi_{m,n-1} \Theta_{m-1,n} &= \mathcal{A} \Phi_{m-1,n} \Theta + \mathcal{C} \Phi_{m-1,n} \Theta_{m,n-1} \\ \mathcal{H} \Phi \Theta_{m,n-1} + \mathcal{C} \Phi_{m-1,n} \Theta_{m,n-1} &= \mathcal{B} \Phi_{m,n-1} \Theta + \mathcal{D} \Phi_{m,n-1} \Theta_{m-1,n} \\ (\mathcal{A}_{m+1,n} \Theta_{m+1,n} + \mathcal{B}_{m,n+1} \Theta_{m,n+1} + \mathcal{C}_{m+1,n} \Theta_{m+1,n-1} + \mathcal{D}_{m,n+1} \Theta_{m-1,n+1}) \Phi + \\ \Theta (\mathcal{A} \Phi_{m-1,n} + \mathcal{B} \Phi_{m,n-1}) &= \mathcal{F} \Theta \Phi \end{aligned}$$

Happily enough the first two equations of can be satisfied by putting

$$\Phi = \Theta, \quad \mathcal{G} = \mathcal{A}, \quad \mathcal{H} = \mathcal{B}, \quad \mathcal{C} = \mathcal{D}$$

while the third one takes form

$$\begin{aligned} \mathcal{A}_{m+1,n} \Theta_{m+1,n} + \mathcal{A} \Theta_{m-1,n} + \mathcal{B}_{m,n+1} \Theta_{m,n+1} + \mathcal{B} \Theta_{m,n-1} + \\ \mathcal{C}_{m+1,n} \Theta_{m+1,n-1} + \mathcal{C}_{m,n+1} \Theta_{m-1,n+1} &= \mathcal{F} \Theta \end{aligned} \quad (44)$$

As a result we obtain Darboux transformation for 7-point self-adjoint scheme (c.f. [16])

Theorem 3 Map $N \mapsto \bar{N}$ given by

$$\begin{bmatrix} \Delta_m(\bar{N}\Theta) \\ \Delta_n(\bar{N}\Theta) \end{bmatrix} = \begin{bmatrix} \mathcal{C}\Theta_{m-1,n}\Theta_{m,n-1} & -\Theta_{m,n-1}(\mathcal{B}\Theta + \mathcal{C}\Theta_{m-1,n}) \\ \Theta_{m-1,n}(\mathcal{A}\Theta + \mathcal{C}\Theta_{m,n-1}) & -\mathcal{C}\Theta_{m-1,n}\Theta_{m,n-1} \end{bmatrix} \begin{bmatrix} \Delta_{-m}\frac{N}{\Theta} \\ \Delta_{-n}\frac{N}{\Theta} \end{bmatrix} \quad (45)$$

is a map from solution space of equation

$$\begin{aligned} & \mathcal{A}_{m+1,n} N_{m+1,n} + \mathcal{A} N_{m-1,n} + \mathcal{B}_{m,n+1} N_{m,n+1} + \mathcal{B} N_{m,n-1} + \\ & \mathcal{C}_{m+1,n} N_{m+1,n-1} + \mathcal{C}_{m,n+1} N_{m-1,n+1} = \mathcal{F} N, \end{aligned} \quad (46)$$

to solution space of equation

$$\begin{aligned} & \bar{\mathcal{A}}_{m+1,n} \bar{N}_{m+1,n} + \bar{\mathcal{A}} \bar{N}_{m-1,n} + \bar{\mathcal{B}}_{m,n+1} \bar{N}_{m,n+1} + \bar{\mathcal{B}} \bar{N}_{m,n-1} + \\ & \bar{\mathcal{C}}_{m+1,n} \bar{N}_{m+1,n-1} + \bar{\mathcal{C}}_{m,n+1} \bar{N}_{m-1,n+1} = \bar{\mathcal{F}} \bar{N}, \end{aligned} \quad (47)$$

where Θ is a solution of equation (46) and the new (bared) fields are related to the old ones by

$$\begin{aligned} \bar{\mathcal{A}}_{m+1,n} &= \frac{\Theta_{m+1,n}\mathcal{A}}{\Theta_{m,n-1}\mathcal{P}}, & \bar{\mathcal{B}}_{m,n+1} &= \frac{\Theta_{m,n+1}\mathcal{B}}{\Theta_{m-1,n}\mathcal{P}}, \\ \bar{\mathcal{C}} &= \frac{\mathcal{C}_{m-1,n-1}\Theta_{m-1,n}\Theta_{m,n-1}}{\Theta_{m-1,n-1}\mathcal{P}_{m-1,n-1}}, \\ \bar{F} &= \Theta(\bar{\mathcal{A}}_{m+1,n}\frac{1}{\Theta_{m+1,n}} + \bar{\mathcal{A}}\frac{1}{\Theta_{m-1,n}} + \bar{\mathcal{B}}_{m,n+1}\frac{1}{\Theta_{m,n+1}} + \bar{\mathcal{B}}\frac{1}{\Theta_{m,n-1}} + \\ & \bar{\mathcal{C}}_{m+1,n}\frac{1}{\Theta_{m+1,n-1}} + \bar{\mathcal{C}}_{m,n+1}\frac{1}{\Theta_{m-1,n+1}}) \end{aligned} \quad (48)$$

and where $\mathcal{P} := \Theta\mathcal{A}\mathcal{B} + \Theta_{m-1,n}\mathcal{C}\mathcal{A} + \Theta_{m,n-1}\mathcal{C}\mathcal{B}$.

Clearly the scheme above admits specification $\mathcal{C} = 0$ (alternatively one can put $\mathcal{A} = 0$ or $\mathcal{B} = 0$) and as result we obtain specification to 5-point self-adjoint scheme (c.f. [16]).

7.3 Discrete Moutard type transformation, a 10-point scheme

The idea is that in the Darboux transformation for the 7-point scheme we considered in the previous subsection one can replace field N with $\Psi_{m+1,n} + \Psi_{m,n+1}$ without affecting leading terms of continuum limit. As a result we get the 10-point scheme

$$\begin{aligned} & (\mathcal{A} + \mathcal{B})\Psi + (\mathcal{A}_{m+1,n} + \mathcal{B}_{m,n+1})\Psi_{m+1,n+1} + \\ & \mathcal{A}\Psi_{m-1,n+1} + \mathcal{B}\Psi_{m+1,n-1} + \mathcal{A}_{m+1,n}\Psi_{m+2,n} + \mathcal{B}_{m,n+1}\Psi_{m,n+2} + \\ & \mathcal{C}_{m+1,n}\Psi_{m+2,n-1} + \mathcal{C}_{m,n+1}\Psi_{m-1,n+2} + \\ & (\mathcal{C}_{m+1,n} - F)\Psi_{m+1,n} + (\mathcal{C}_{m,n+1} - F)\Psi_{m,n+1} = 0 \end{aligned} \quad (49)$$

which is no longer self-adjoint but if the function Θ satisfies the equation (49)

$$\begin{aligned} & (\mathcal{A} + \mathcal{B})\Theta + (\mathcal{A}_{m+1,n} + \mathcal{B}_{m,n+1})\Theta_{m+1,n+1} + \\ & \mathcal{A}\Theta_{m-1,n+1} + \mathcal{B}\Theta_{m+1,n-1} + \mathcal{A}_{m+1,n}\Theta_{m+2,n} + \mathcal{B}_{m,n+1}\Theta_{m,n+2} + \\ & \mathcal{C}_{m+1,n}\Theta_{m+2,n-1} + \mathcal{C}_{m,n+1}\Theta_{m-1,n+2} + \\ & + (\mathcal{C}_{m+1,n} - F)\Theta_{m+1,n} + (\mathcal{C}_{m,n+1} - F)\Theta_{m,n+1} = 0 \end{aligned} \quad (50)$$

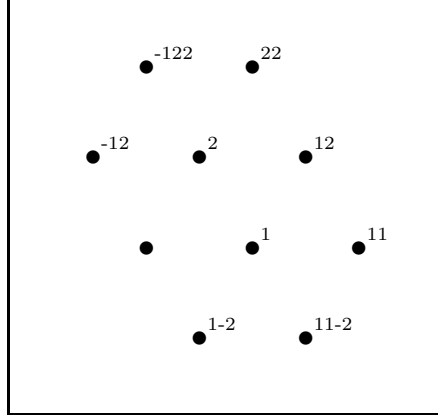


Figure 4: 10-point scheme

then the function

$$\Phi = \Theta_{m+1,n} + \Theta_{m,n+1}$$

satisfies the equation formally adjoint to eq. (49)

$$\begin{aligned} & (\mathcal{A} + \mathcal{B})\Phi + (\mathcal{A}_{m,n-1} + \mathcal{B}_{m-1,n})\Phi_{m-1,n-1} + \mathcal{A}_{m+1,n-1}\Phi_{m+1,n-1} + \\ & \mathcal{B}_{m-1,n+1}\Phi_{m-1,n+1} + \mathcal{A}_{m-1,n}\Phi_{m-2,n} + \mathcal{B}_{m,n-1}\Phi_{m,n-2} + \\ & \mathcal{C}_{m-1,n+1}\Phi_{m-2,n+1} + \mathcal{C}_{m+1,n-1}\Phi_{m+1,n-2} + \\ & + (\mathcal{C} - F_{m-1,n})\Phi_{m-1,n} + (\mathcal{C} - F_{m,n-1})\Phi_{m,n-1} = 0 \end{aligned} \quad (51)$$

The scheme is proper for solving the initial-boundary value problem we discussed in section 2.

We finally receive transformation

$$\begin{aligned} & \frac{\Delta_n((\bar{\Psi}_{m+1,n} + \bar{\Psi}_{m,n+1})(\Theta_{m+1,n} + \Theta_{m,n+1}))}{(\Theta + \Theta_{m-1,n+1})} = \\ & [\mathcal{A}(\Theta_{m+1,n} + \Theta_{m,n+1}) + \mathcal{C}(\Theta + \Theta_{m+1,n-1})] \Delta_{-m} \frac{\Psi_{m+1,n} + \Psi_{m,n+1}}{\Theta_{m+1,n} + \Theta_{m,n+1}} - \\ & \mathcal{C}(\Theta + \Theta_{m+1,n-1}) \Delta_{-n} \frac{\Psi_{m+1,n} + \Psi_{m,n+1}}{\Theta_{m+1,n} + \Theta_{m,n+1}} \quad (52) \\ & \frac{\Delta_m((\bar{\Psi}_{m+1,n} + \bar{\Psi}_{m,n+1})(\Theta_{m+1,n} + \Theta_{m,n+1}))}{(\Theta + \Theta_{m+1,n-1})} = \\ & - [\mathcal{B}(\Theta_{m+1,n} + \Theta_{m,n+1}) + \mathcal{C}(\Theta + \Theta_{m-1,n+1})] \Delta_{-n} \frac{\Psi_{m+1,n} + \Psi_{m,n+1}}{\Theta_{m+1,n} + \Theta_{m,n+1}} - \\ & \mathcal{C}(\Theta + \Theta_{m-1,n+1}) \Delta_{-m} \frac{\Psi_{m+1,n} + \Psi_{m,n+1}}{\Theta_{m+1,n} + \Theta_{m,n+1}} \end{aligned}$$

8 Conclusions and perspectives

We end the paper with some conclusions and propositions for further development.

The generalization of results of this paper to multi-dimension can be given where set of difference operators Δ_i , $i = 1, \dots, N$ is replaced by a set operators

D_i , $i = 1, \dots, N$ that are linear commuting combinations of Δ_i . The same reasoning holds as far as Laplace transformations are concerned. The 6-point scheme admit factorization $[(M_1 T_1 + N_1 T_2 + X_1)(M_2 T_1 + N_2 T_2 + X_2) + H]\psi = 0$ (where T_1 and T_2 are shifts in m and n direction respectively. so it can be used to develop theory of Laplace transformations for difference equations [31, 32, 33, 34, 24, 39].

We would like to mention that on this level of generality showing q -difference analogue discrete schemes we just introduced is straightforward [35].

Four propositions of further developments are in order. First, generalization of quadratic and symmetric reductions to 6-point scheme (it will be given in forthcoming papers). Second, simple idea by professor Decio Levi that not only operators $\Delta_m \Delta_n \Delta_{-m} \Delta_{-n}$ can be of importance e.g. one can try to use $T_1 - T_2$ and $T_1 T_2 - 1$ operators instead. Third, to investigate the role of just presented transformations in the difference geometry. Finally and the most importantly, deriving hierarchies of nonlinear equations associated with all the equations presented in the paper.

Since we concentrated here on discretizations of hyperbolic equations we end the article with three comments on star schemes – proper discretizations of elliptic differential equations. Firstly, let us notice that if we substitute $\psi = \phi + \phi_{m+1,n} + \phi_{m,n+1}$ into the 6-point scheme we will obtain star-like difference scheme, which can serve as star like discretization of eq. (6) and which needs further studies.

Secondly, it is remarkable that star like (or cross like) operators such as 7-point scheme or 5-point scheme appeared in the integrable literature occasionally [36, 32, 37, 33, 38, 39]. But almost none of the results were used to obtain solutions of nonlinear integrable systems. The only exceptions are works concerning discrete time Toda chains [40, 41, 42, 43, 44, 45] which are a nonlinear 5-point scheme itself and the work [16] the result of which were used to obtain solutions of generalization of Toda chain to two dimensional lattice [46]. In other words theory of integrable difference systems based on the schemes other than 4-point schemes is still in its infancy.

Finally, the relationship of integrable systems on quad-graphs with equations on stars the discrete time Toda type lattices are established [12, 13, 14, 15] and we would like to refer to the relationship as to sub-lattice approach. It is not clear under what circumstances the sub-lattice approach does not destroy integrability. However in the paper [47] it was proved that for discrete lattices governed by Moutard equation integrability features like existence of Darboux transformations are inherited by a sub-lattice.

Acknowledgments The main part of the paper has been presented as a poster during the conference Symmetries and Integrability of Difference Equations SIDE VI June 19-24, 2004 Helsinki. I would like to express my thankfulness to organizers of the SIDE VI conference for the support that enable me to take part in the fruitful conference. I would like to acknowledge Paolo Maria Santini for motivating me to write the paper, Jan Cieřliński for indicating me that Combescure transformation is fundamental in Jonas's fundamental transformation and Frank Nijhoff for the literature guidance.

References

- [1] Moutard, Th.F.: Sur la construction des équations de la forme $\frac{1}{z} \frac{\partial^2 z}{\partial x \partial y} = \lambda(x, y)$ qui admettent une integrale général explicite. J. Ec. Pol. **45**, 1 (1878)
- [2] Darboux, G.: Sur une proposition relative aux équations linéaires. C. R. Acad. Sci. Paris **94** 1456-1459 (1882)
- [3] Matveev, V.B. and Sale, M.A.: *Darboux transformations and Solitons*. Berlin: Springer, 1991
- [4] Jonas, H.: Über die Transformation der konjugierten Systeme und über den gemeinsamen Ursprung der Bianchischen Permutabilitätstheoreme. Berlin Sitzungsber. **XIV**, 96-118 (1915).
- [5] Sauer, R.: *Differenzgeometrie*, Berlin: Springer, 1970
- [6] Eisenhart, L.P.: *Transformation of Surfaces*. Princeton, N. J.: Princeton Univ. Press, 1923
- [7] Kac, V.G., and van de Leur, J.W.: The n-component KP hierarchy and representation theory J. Math. Phys. **44**, 3245-3293 (2003)
- [8] Doliwa, A.: On tau-Function of Conjugate Net. J. Nonlinear Math. Phys. **12** sup.1, 244–252 (2005)
- [9] Doliwa, A. and Santini, P.M.: Multidimensional quadrilateral lattices are integrable. Phys. Lett. A **233**, 365–372 (1997)
- [10] Mañas, M., Doliwa, A. and Santini, P.M.: Darboux transformations for multidimensional quadrilateral lattices .1. Phys. Lett. A **232**, 99–105 (1997)
- [11] Doliwa, A., Santini, P.M. and Mañas, M.: Transformations of quadrilateral lattices. J. Math. Phys. **41**, 944–990 (2000)
- [12] Adler, V.E.: Discrete equations on planar graphs. J. Phys. A: Math. Gen. **34**, 10453–10460 (2001)
- [13] Bobenko, A.I. and Suris, Yu.B.: Integrable systems on quad-graphs. Int. Math. Res. Notices **11**, 573–611 (2002)
- [14] Bobenko, A.I. and Suris, Yu.B.: Integrable noncommutative equations on quad-graphs. The consistency approach. Lett. Math. Phys. **61**, 241–254 (2002)
- [15] Adler, V.E., Bobenko, A.I. and Suris, Yu.B.: Classification of integrable equations on quad-graphs. The consistency approach. Comm. Math. Phys. **233**, 513–543 (2003)
- [16] Nieszporski, M., Santini, P.M. and Doliwa, A.: Darboux transformations for 5-point and 7-point self-adjoint schemes and an integrable discretization of the 2D Schrödinger operator. Phys. Lett. A **323**, 241–250 (2004)

- [17] Hildebrand, F.B.: *Finite Difference Equations and Simulations*. Englewood Cliffs: PrenticeHall, 1968
- [18] Laplace, P.S.: Recherches sur le Calcul intégral aux différences partielles, Mémoires de Mathématique et de Physique de l'Académie des Sciences 341–403 (1773)
- [19] Darboux, G.: *Lecons sur la Théorie Générale des Surfaces*. Paris: Gauthier-Villars, 1887
- [20] Liu, Q.P. and Manas, M.: Discrete Levy transformations and Casorati determinant solutions of quadrilateral lattices. Phys. Lett. A **239**, 159–166 (1998)
- [21] Willox, R., Ohta, Y., Gilson, C.R., Tokihiro, T. and Satsuma, J.: Quadrilateral lattices and eigenfunction potentials for N-component KP hierarchies. Phys. Lett. A **252**, 163–172 (1999)
- [22] Manas, M.: Fundamental transformations for quadrilateral lattices: first potentials and tau-functions, symmetric and pseudo-Egorov reductions. J. Phys. A-Math. Gen. **34**, 10413–10421 (2001)
- [23] Nimmo, J.J.C. and Schief, W.K.: Superposition principles associated with the Moutard transformation: an integrable discretization of a 2+1-dimensional sine-Gordon system. Proc. R. Soc. London A **453**, 255–279 (1997)
- [24] Nieszporski, M.: A Laplace ladder of discrete Laplace equations. Theor. Math. Phys. **133**, 1576–1584 (2002)
- [25] Cieslinski, J., Doliwa, A. and Santini, P.M.: The integrable discrete analogues of orthogonal coordinate systems are multi-dimensional circular lattices. Phys. Lett. A 235 (1997) 480–488.
- [26] B.G. Konopelchenko, W.K. Schief, Three-dimensional integrable lattices in Euclidean spaces: conjugacy and orthogonality. P. Roy. Soc. Lond. A Mat. **454**, 3075–3104 (1998)
- [27] Liu, Q.P. and Manas, M.: Superposition formulae for the discrete Ribaucour transformations of circular lattices. Phys. Let. A **249**, 424–430 (1998)
- [28] Doliwa, A.: Quadratic Reductions of Quadrilateral Lattices. J. Geom. Phys. **30**, 169–186 (1999)
- [29] Schief, W. K.: Finite analogues of Egorov-type coordinates, unpublished, talk given at the Workshop:Nonlinear systems, solitons and geometry, Oberwolfach, October 1997.
- [30] Doliwa, A. and Santini, P.M.: The symmetric, D-invariant and Egorov reductions of the quadrilateral lattice. J. Geom. Phys. **36**, 60–102 (2000)

- [31] Doliwa, A.: Geometric discretisation of the Toda system. *Phys. Lett. A* **234**, 187–192 (1997) A. Doliwa, *Phys. Lett. A*, 234, 187 (1997)
- [32] Novikov, S.P.: Algebraic properties of two-dimensional difference operators. *Russian Math. Surveys* **52**, 226–227 (1997)
- [33] Novikov, S.P. and Dynnikov, I.A.: Discrete spectral symmetries of low-dimensional differential operators and difference operators on regular lattices and two-dimensional manifolds. *Russ. Math. Surveys* **52**, 1057–1116 (1997)
- [34] Adler, V.E. and Startsev, S.Y.: Discrete analogues of the Liouville equation. *Theor. Math. Phys.* **121**, 1484–1495 (1999)
- [35] Małkiewicz, P. and Nieszporski, M.: Darboux transformations for q-discretizations of 2D second order differential equations. to appear in *J. Nonlinear Math. Phys.*
- [36] Krichever, I.M.: Two-dimensional periodic difference operators and algebraic geometry. *Doklady Akademii Nauk SSSR* **285**, 31–6 (1985). *Soviet Math. Dokl.* **32**, 623–627 (1985)
- [37] Nijhoff, F.W. and Papageorgiou, V.G.: Similarity reductions of integrable lattices and discrete analogs of the painleve-II equation. *Phys. Lett. A* **153**, 337–344 (1991)
- [38] Nijhoff, F.W. and Walker, A.J.: The discrete and continuous Painleve VI hierarchy and the Garnier systems. *Glasgow Math. J.* **43A**, 109–123 (2001)
- [39] Oblomkov, A.A. and Penskoi A.V.: Laplace transformations and spectral theory of two-dimensional semidiscrete and discrete hyperbolic Schrodinger operators *Int. Math. Res. Notices* **18**, 1089–1126 (2005)
- [40] Hirota, R.: Non-linear partial difference equations .2. discrete time Toda equation. *J. Phys. Soc. Japan* **43**, 2074–2078 (1977)
- [41] Hirota, R.: Non-linear partial difference equations .4. Backlund transformation for discrete time Toda equation. *J. Phys. Soc. Japan* **45**, 321–332 (1978)
- [42] Suris, Yu.B.: Discrete-time generalized Toda lattices: complete integrability and relation with relativistic Toda lattices. *Phys. Lett. A* **145**, 113–19 (1990)
- [43] Suris, Yu.B.: Generalized Toda chains in discrete times. *Lenningrad Math. J.* **2**, 339–52 (1991)
- [44] Suris, Yu.B.: Algebraic structure of discrete-time and relativistic Toda lattices. *Phys. Lett. A* **156**, 467–74 (1991)

- [45] Suris, Yu.B.: On some integrable systems related to the Toda lattice. J. Phys. A: Math. Gen. **30**, 2235-2249 (1997)
- [46] Santini, P.M., Nieszporski, M. and Doliwa, A.: Integrable generalization of the Toda law to the square lattice. Phys. Rev. E **70**, Art. No. 056615 (2004)
- [47] Doliwa, A., Grinevich, P., Nieszporski, M. and Santini, P.M.: Integrable lattices and their sublattices: from the discrete Moutard (discrete Cauchy-Riemann) equation to the self-adjoint 5-point scheme, nlin.SI/0410046.